Journal of Statistical Physics, Vol. 54, Nos. 3/4, 1989

An Exact Representation of the Space–Time Characteristic Functional of Turbulent Navier–Stokes Flows with Prescribed Random Initial States and Driving Forces

Taylan Alankus[1]

Received March 8, 1988; final October 20, 1988

By adopting a formal operator viewpoint, the space–time characteristic functional associated with Navier–Stokes turbulence is expressed in terms of a linear operator acting on the space of functionals. Obtained by a simple similarity transformation of the local translation operator generated by the nonlinear terms in the Navier–Stokes equation, this operator is unitary with respect to the formal scalar product of functionals. The equivalence of this operator representation to the functional integral representation of Rosen is shown and, for Gaussian initial velocity and external force fields, some consequences of this representation are presented.

KEY WORDS: Fluid turbulence; the Navier–Stokes equation; space–time characteristic functional; formally unitary operator; functional integration.

1. INTRODUCTION

The basic assumption of turbulence theory is that the statistical dynamics of a turbulent flow is completely determined by the joint probability measure associated with random hydrodynamic fields whose realizations satisfy the mass, momentum, and energy equations for prescribed initial and boundary conditions. For incompressible fluids with uniform density and viscosity the spatial (Eulerian) velocity field $u(x, t)$ alone is sufficient to specify the fluid motion. Thus, an explicit determination of the characteristic functional of $u(x, t)$, which incorporates all the finite-dimensional

[1] Department of Mechanical Engineering and Laboratory for Planetary Atmospheres Research, SUNY at Stony Brook, Stony Brook, New York 11794.

0022-4715/89/0200-0859$06.00/0

statistical observables associated with a turbulent fluid, constitutes a central problem for the deductive theory of turbulence. The Navier–Stokes dynamics induces a linear evolution equation (Hopf's equation[1]) for the space characteristic functional of $u(x, t)$ which simply expresses the preservation of normalization property of probability measure under this dynamics. The exact general solution of this equation for a given initial statistical state can be formally represented as a functional integral; however, the deduction of relevant physical information from this representation is still an unsolved problem.

Similarly, the space–time characteristic functional of $u(x, t)$ satisfies a linear equation and also admits a general integral representation. These integral representations can also be derived directly from the Navier–Stokes equation without referring to the functional differential equations. For boundary-free fluids which are not subject to random external forces, these representations were presented first by Rosen.[2,3] In the special cases of zero viscosity or negligible nonlinear interactions, more explicit exact expressions for the space and space–time characteristic functionals have also been derived under particular statistical restrictions by various researchers.[4–6] The functional integral representations associated with the Navier–Stokes turbulence can essentially be regarded as linear mappings between particular characteristic functionals. Since the Navier–Stokes dynamics is not measure preserving in the phase space corresponding to $u(x, t)$ due to viscous dissipation, a unitary linear operator cannot be associated with the evolution of fluid turbulence. However, by adopting an operator viewpoint, it is possible to express the space–time characteristic functional of $u(x, t)$ exactly, and explicitly, in terms of a functional linear transformation which is formally unitary with respect to a particular scalar product in the space of functionals, as shown in this paper. This expression is valid for any admissible prescribed statistics of the initial velocity and external force fields and can be interpreted as a perturbation expansion in powers of a particular Reynolds number.

Here only the fluids without boundaries are considered, although the boundary conditions and geometries can be incorporated into the formalism without essential changes. The universal, stationary, small-scale characteristics of a turbulent flow can be specified without dealing with boundaries and initial statistics by working with a boundary-free and externally driven fluid. In Section 2, a formal series representation for the linear operator which connects the space–time characteristic functionals of linear and actual turbulent flows is derived through a simple similarity transformation. The connections to Rosen's functional integral representation[3] are also shown in this section. Section 3 contains, for Gaussian initial velocity and driving force fields, some simple consequences of this operator representation.

2. THE SPACE–TIME OPERATOR FOR TURBULENCE STATISTICS

The laminar and turbulent motions of boundary-free incompressible fluids are specified by the Navier–Stokes equation[7]:

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)\,u-\nu\Delta u+\nabla p=f, \qquad u_{|t=t_0}=u_0(x), \qquad \nabla u_0=0 \tag{1}$$

where $x=(x^1, x^2, x^3)$ is the (Cartesian) position vector of a point in Euclidean space R^3, $u\equiv u(x, t)$ is the spatial velocity field, $p\equiv p(x, t)$ is the (modified) pressure, ν is the kinematic viscosity coefficient, and $f\equiv f(x, t)$ is the external volume force density, which can be prescribed as divergenceless without loss of generality. The solution set of (1) consists of solenoidal vectors only, since the condition $\nabla u_0=0$ implies $\nabla u=0$ for $t>t_0$. The statistical turbulence problem associated with (1) is simply the determination of the space–time characteristic functional

$$Z(\eta)=\int e^{i(\eta,u)}\,d\mu(u)\equiv\left\langle \exp\left\{i\int dz\,\eta(z)\cdot u(z)\right\}\right\rangle \tag{2}$$

from the prescribed functionals

$$Z_i(\theta)=\int e^{i[\theta,u_0]}\,d\mu_i(u_0)\equiv\left\langle \exp\left\{i\int dx\,\theta(x)\cdot u_0(x)\right\}\right\rangle \tag{3}$$

and

$$Z_e(\eta)=\int e^{i(\eta,f)}\,d\mu_e(f)\equiv\left\langle \exp\left\{i\int dz\,\eta(z)\cdot f(z)\right\}\right\rangle \tag{4}$$

where $z\equiv(x, t)$, $\eta(z)$ and $\theta(x)$ denote the test vector fields, the dot is the usual vector product, and the x and z integrations are over entire R^3 and $R^3\times[t_0, \infty)$, respectively. The pressure field can be eliminated from (1) by applying the incompressibility condition and solving the resulting Poisson equation and one can recast (1) into an integral form with the help of heat kernel associated with R^3 as

$$u^\alpha(z)=v^\alpha(z)+\int dz_1\,C_{\alpha\beta\gamma}(z-z_1)\,u^\beta(z_1)\,u^\gamma(z_1) \tag{5}$$

where

$$v(z)=\int dx_1\, G(x-x_1, t-t_0)\, u_0(x_1)+\int dz_1\, G(z-z_1)\, f(z_1) \tag{6}$$

$$G(z)=\begin{cases}(4\pi\nu)^{-3/2}\exp(-|x|^2/4\nu t), & t>0\\ 0, & t<0\end{cases} \tag{7}$$

$$C_{\alpha\beta\gamma}(z)=-\frac{1}{2}\left(\delta_{\alpha\beta}\frac{\partial}{\partial x^\gamma}+\delta_{\alpha\gamma}\frac{\partial}{\partial x^\beta}\right)G(z)-\frac{\partial^3}{\partial x^\alpha \partial x^\beta \partial x^\gamma}$$

$$\times\int dx_1\,\frac{G(x_1, t)}{4\pi\,|x-x_1|} \tag{8}$$

for $t>t_0$, and $C_{\alpha\beta\gamma}(z)$ vanishes for $t\leqslant t_0$ [further properties of $C_{\alpha\beta\gamma}(z)$ can be found in ref. 8].

The velocity field $v(z)$ is solenoidal and its characteristic functional

$$Z_0(\eta)=\int e^{i(\eta,v)}\, d\mu_0(v)\equiv\left\langle \exp\left\{i\int dz\, \eta(z)\cdot v(z)\right\}\right\rangle \tag{9}$$

is completely determined in terms of independently prescribed statistics of $u_0(x)$ and $f(z)$ as

$$Z_0(\eta)=Z_i\left(\int dz_1\, G(x_1-x, t_1-t_0)\,\eta(z_1)\right)Z_e\left(\int dz_1\, G(z_1-z)\,\eta(z_1)\right) \tag{10}$$

Thus, the Navier–Stokes turbulence problem, as stated above, reduces to the explicit determination of $Z(\eta)$ in terms of $Z_0(\eta)$. To this end, let us write Eq. (5) as

$$u(z)=v(z)+\lambda Q(u; z) \tag{11}$$

where the quadratic functional $Q(u;z)=(Q_1, Q_2, Q_3)$ represents the integral term in (5) and the real parameter λ tags this term. Now we can treat (11) as a one-parameter nonlinear mapping between solenoidal vector fields, so that by restricting the admissible sets of $u_0(x)$ and $f(z)$ (and the time interval if necessary) to those for which Eq. (11) has a unique solution $u(z)$ for a given $v(z)$, one may introduce the mapping $T: u\rightarrow v=u-\lambda Q(u)\equiv T(u)$ and its inverse $T^{-1}: v\rightarrow u\equiv T^{-1}(v)$ which solves the Navier–Stokes equation. In turn, T induces a linear transformation $\hat{T}$ on the space of functionals of vector fields u such that, by definition, $(\hat{T}F)(u)=F(T(u))\equiv F(u-\lambda Q(u))$ for an arbitrary functional $F(u)$. Clearly, for $\lambda=0$ T and $\hat{T}$ are identity mappings in their corresponding spaces. The functional determinant $\det(T)$ of mapping T, which can be formally expressed as $\exp\{\mathrm{Tr}\ln(1-\lambda\,\delta Q/\delta u)\}$, is equal to 1 since the linear operator corresponding to kernel

$$\delta Q_\alpha(z_1)/\delta u^\beta(z_2) = 2C_{\alpha\beta\gamma}(z_1 - z_2)\, u^\gamma(z_2)$$

and its powers are traceless.[3] This implies that the operator $\hat{T}$ is formally unitary with respect to the scalar product of two functionals F and G defined by

$$\langle F, G\rangle = \int d(u)\, \overline{F(u)}\, G(u) \tag{12}$$

where the overbar denotes complex conjugation and the formal measure $d(u)$ corresponds to the space–time lattice approximation of vector fields by step functions,[6] namely

$$d(u) = \lim \prod_j \left(\frac{\mathrm{Vol}(\Omega_j)}{2\pi}\right)^{3/2} du_j \tag{13}$$

In (13), $\mathrm{Vol}(\Omega_j)$ is the volume of space–time cell Ω_j in which $u(z)$ is approximated by

$$u_j = [\mathrm{Vol}(\Omega_j)]^{-1} \int_{\Omega_j} dz\, u(z)$$

Since $d(T(u)) = |\det(T)|\, d(u) = d(u)$, we have

$$\langle F, \hat{T}G\rangle = \langle \hat{T}^{-1}F, G\rangle \tag{14}$$

which evidently implies $\langle \hat{T}F, \hat{T}G\rangle = \langle F, G\rangle$. $\hat{T}$ has a simple formal series representation which can be regarded as an ordered exponential operator, namely

$$\begin{aligned} \hat{T} &= \exp_R\left\{-\lambda\left(Q(u), \frac{\delta}{\delta u}\right)\right\} \\ &= \sum_{n\geq 0} \frac{(-\lambda)^n}{n!} \int \cdots \int dz_1 \cdots dz_n\, Q_{\alpha_1}(u; z_1) \\ &\quad \times \cdots Q_{\alpha_n}(u; z_n) \frac{\delta}{\delta u^{\alpha_1}(z_1)} \cdots \frac{\delta}{\delta u^{\alpha_n}(z_n)} \end{aligned} \tag{15}$$

and a direct calculation also leads to the representation of the inverse $\hat{T}^{-1}$ of $\hat{T}$ as

$$\begin{aligned}\hat{T}^{-1} &= \exp_L\left\{\lambda\left(\frac{\delta}{\delta u}, Q(u)\right)\right\} \\ &= \sum_{n\geqslant 0}\frac{\lambda^n}{n!}\int\cdots\int dz_1\cdots dz_n\frac{\delta}{\delta u^{\alpha_1}(z_1)}\cdots\frac{\delta}{\delta u^{\alpha_n}(z_n)} \\ &\quad\times Q_{\alpha_1}(u; z_1)\cdots Q_{\alpha_n}(u; z_n)\end{aligned} \tag{16}$$

Now, if we formally set $d\mu(u) = P(u)\, d(u)$ and $d\mu_0(u) = P_0(u)\, d(u)$, it follows from

$$P(u) = |\det(T)|\; P_0(T(u)) = (\hat{T}P_0)(u)$$

and the definitions of $Z(\eta)$ and $Z_0(\eta)$ that

$$Z(\eta) = (\hat{F}^{-1}\hat{T}\hat{F}Z_0)(\eta) \tag{17}$$

where $\hat{F}$ is the Fourier operator defined by

$$(\hat{F}F)(\eta) = \int d(\eta')\, e^{-i(\eta,\eta')}F(\eta') \tag{18}$$

and $\hat{F}^{-1}$ is the inverse of $\hat{F}$ given by

$$(\hat{F}^{-1}F)(\eta) = \int d(\eta')\, e^{i(\eta,\eta')}F(\eta') \tag{19}$$

The operator $\hat{F}$ is unitary with respect to the product (12); consequently, the statistical solution operator

$$\hat{S} \equiv \hat{F}^{-1}\hat{T}\hat{F} \equiv \hat{F}^{-1}e_R^{-\lambda(Q(\eta),\delta/\delta\eta)}\hat{F} \tag{20}$$

for $Z(\eta)$ is also unitary for all values of λ. In this sense, $Z(\eta)$ and $Z_0(\eta)$, representing nonlinear and linear turbulent flow statistics, respectively, are unitary transformations of each other. An explicit representation of operator $\hat{S}$, a similarity transformation of $\hat{T}$, is provided by the following proposition:

If $\hat{T}$ and $\hat{F}$ are defined by $(\hat{T}F)(\eta) = F(\eta - \lambda Q(\eta))$ and (18), respectively, and $\det(T) = \det(1 - \lambda\, \delta Q/\delta\eta) = 1$, then $\hat{S} = \hat{F}^{-1}\hat{T}\hat{F}$ has the representation

$$\hat{S} = \exp_L \left\{ i\lambda \left(Q\left(\frac{\delta}{i\delta\eta}\right), \eta \right) \right\}$$
$$= \sum_{n \geqslant 0} \frac{(i\lambda)^n}{n!} \int \cdots \int dz_1 \cdots dz_n \, Q_{\alpha_1}\left(\frac{\delta}{i\delta\eta}; z_1\right) \cdots Q_{\alpha_n}\left(\frac{\delta}{i\delta\eta}; z_n\right)$$
$$\times \eta^{\alpha_1}(z_1) \cdots \eta^{\alpha_n}(z_n) \tag{21}$$

This can be verified directly by using definitions (15) and (18) and the simple properties of functional integrals; it can also be shown that the operator

$$\hat{S}^{-1} = \exp_R \left\{ -i\lambda \left(\eta, Q\left(\frac{\delta}{i\delta\eta}\right) \right) \right\}$$
$$= \sum_{n \geqslant 0} \frac{(-i\lambda)^n}{n!} \int \cdots \int dz_1 \cdots dz_n \, \eta^{\alpha_1}(z_1) \cdots \eta^{\alpha_n}(z_n)$$
$$\times Q_{\alpha_1}\left(\frac{\delta}{i\delta\eta}; z_1\right) \cdots Q_{\alpha_n}\left(\frac{\delta}{i\delta\eta}; z_n\right) \tag{22}$$

satisfies $\hat{S}\hat{S}^{-1} = \hat{S}^{-1}\hat{S} = \hat{1}$ for all λ, where $\hat{1}$ is the identity operator. By substituting (21) into (17), we get the final expression for $Z(\eta)$ in terms of $Z_0(\eta)$ as

$$Z(\eta) = \left[\exp_L \left\{ i\lambda \left(Q\left(\frac{\delta}{i\delta\eta}\right), \eta \right) \right\} Z_0 \right](\eta) \tag{23}$$

or, explicitly,

$$Z(\eta) = Z_0(\eta) + \sum_{n \geqslant 1} \frac{(-i\lambda)^n}{n!} \int \cdots \int dz_1 \, dz'_1 \cdots dz_n \, dz'_n \left\{ \prod_{k=1}^{n} C_{\alpha_k \beta_k \gamma_k}(z_k - z'_k) \right\}$$
$$\times \left\{ \frac{\delta^2}{\delta\eta^{\beta_1}(z'_1)\, \delta\eta^{\gamma_1}(z'_1)} \cdots \frac{\delta^2}{\delta\eta^{\beta_n}(z'_n)\, \delta\eta^{\gamma_n}(z'_n)} \right\}$$
$$\times \left\{ \prod_{k=1}^{n} \eta^{\alpha_k}(z_k) \right\} Z_0(\eta) \tag{24}$$

This λ expansion is essentially in powers of a particular Reynolds number which can be interpreted as an interaction constant. Specifically, in terms of the dimensionless variables x/l_0, u/v_0, and tv/l_0^2 we have $\lambda = l_0 v_0/v \equiv Re$. Here l_0 and v_0 can be, for example, the integral and rms velocity scales associated with $v(z)$ or $u_0(x)$, respectively. The more refined Reynolds number expansion schemes, as well as the direct expansion (24), are likely to yield unsatisfactory convergence and analyticity properties for physically relevant *Re*'s; the careless truncations of (24) may also lead to unphysical behavior.[9] However, (24) provides a standard which can be used to assess the validity of a statistical approximation in turbulence theory.

Now we define the kernel (i.e., "matrix element") $S(\eta, \eta')$ of $\hat{S}$ as

$$S(\eta, \eta') = \hat{S}\delta(\eta - \eta') \equiv \hat{S} \int d(\omega)\, e^{i(\omega, \eta - \eta')} \tag{25}$$

where $\hat{S}$ acts on the η dependence of the delta functional[6] only. This, from (21), directly gives

$$S(\eta, \eta') = \int d(\omega)\, e^{i(\omega, \eta - \eta') + i\lambda(\eta', Q(\omega))} \tag{26}$$

The inverse kernel $S^{-1}(\eta, \eta')$ corresponding to $\hat{S}^{-1}$, by definition, satisfies

$$\int d(\eta'')\, S^{-1}(\eta, \eta'')\, S(\eta'', \eta') = \int d(\eta'')\, S(\eta, \eta'')\, S^{-1}(\eta'', \eta') = \delta(\eta - \eta') \tag{27}$$

and is easily determined from the unitarity property $S^{-1}(\eta, \eta') = \overline{S(\eta', \eta)}$ as

$$S^{-1}(\eta, \eta') = \int d(\omega)\, e^{i(\omega, \eta - \eta') - i\lambda(\eta, Q(\omega))} \tag{28}$$

By substituting (26) into the relation

$$Z(\eta) = (\hat{S}Z_0)(\eta) = \int d(\eta')\, S(\eta, \eta')\, Z_0(\eta') \tag{29}$$

we obtain the integral form of (24) as

$$Z(\eta) = \iint d(\eta')\, d(\eta'')\, Z_0(\eta')\, e^{i(\eta'', \eta - \eta') + i\lambda(\eta', Q(\eta''))} \tag{30}$$

which, for the case of $f(z) \equiv 0$, is equivalent to that of Rosen.[3] [Note that the Gaussian η'' integration in (30) can be formally performed; this has been done by Rosen, whose expression also incorporates an undetermined function to account the possible nonuniqueness of solutions of Eq. (1)].

Finally, for very small Reynolds numbers one may write

$$\hat{S} \cong 1 - i\lambda\hat{H}, \qquad \hat{S}^{-1} \cong 1 + i\lambda\hat{H} \tag{31}$$

where the operator $\hat{H}$ is formally Hermitian (self-adjoint) with respect to (12), that is, $\langle F, \hat{H}G\rangle = \langle \hat{H}F, G\rangle$ for arbitrary functionals F and G, and is explicitly given by

$$\begin{aligned}\hat{H} &= \iint dz_1\, dz_2\, C_{\alpha\beta\gamma}(z_1 - z_2)\left(\frac{\delta^2}{\delta\eta^\beta(z_2)\,\delta\eta^\gamma(z_2)}\right)\eta^\alpha(z_1)\\ &= \iint dz_1\, dz_2\, C_{\alpha\beta\gamma}(z_1 - z_2)\,\eta^\alpha(z_1)\left(\frac{\delta^2}{\delta\eta^\beta(z_2)\,\delta\eta^\gamma(z_2)}\right)\end{aligned} \tag{32}$$

3. GAUSSIAN INITIAL STATES AND DRIVING FORCES

To illustrate some consequences of (24) briefly, let us prescribe $u_0(x)$ and $f(z)$ as Gaussian random fields with zero means and covariances $\langle u_0^\alpha(x_1)\, u_0^\beta(x_2)\rangle \equiv M_{\alpha\beta}(x_1, x_2)$ and $\langle f^\alpha(z_1)\, f^\beta(z_2)\rangle \equiv F_{\alpha\beta}(z_1, z_2)$, respectively. It follows from the linearity of $v(z)$ in $u_0(x)$ and $f(z)$ that $v(z)$ is also Gaussian with zero mean. Thus, from (10) we have

$$\begin{aligned}Z_0(\eta) \equiv Z_G(\eta) &= \exp\left[-\frac{1}{2}(\eta, A\eta)\right]\\ &\equiv \exp\left\{-\frac{1}{2}\iint dz_1\, dz_2\, A_{\alpha\beta}(z_1, z_2)\,\eta^\alpha(z_1)\,\eta^\beta(z_2)\right\}\end{aligned} \tag{33}$$

where $A_{\alpha\beta}(z_1, z_2)$ is the symmetric and positive-definite (nonnegative) covariance of $v(z)$ which is determined by

$$\begin{aligned}A_{\alpha\beta}(z_1, z_2) &= \iint dx_3\, dx_4\, M_{\alpha\beta}(x_3, x_4)\, G(x_1 - x_3, t_1 - t_0)\, G(x_2 - x_4, t_2 - t_0)\\ &\quad + \iint dz_3\, dz_4\, F_{\alpha\beta}(z_3, z_4)\, G(z_1 - z_3)\, G(z_2 - z_4)\end{aligned} \tag{34}$$

In terms of the measures μ and $\mu_0 \equiv \mu_G$, (33) implies

$$\frac{d\mu}{d\mu_G}(u) = \exp\left\{\lambda(Q(u), A^{-1}u) - \frac{\lambda^2}{2}(Q(u), A^{-1}Q(u))\right\} \tag{35}$$

where $A_{\alpha\beta}^{-1}(z_1, z_2)$ is the inverse of $A_{\alpha\beta}(z_1, z_2)$ which satisfies

$$\begin{aligned}&\int dz_3\, A_{\alpha\gamma}^{-1}(z_1, z_3)\, A_{\gamma\beta}(z_3, z_2)\\ &\qquad = \int dz_3\, A_{\alpha\gamma}(z_1, z_3)\, A_{\gamma\beta}^{-1}(z_3, z_2) = \delta_{\alpha\beta}\delta(z_1 - z_2)\end{aligned} \tag{36}$$

and

$$(Q(u), A^{-1}u) \equiv \iiint dz_1\, dz_2\, dz_3\, A^{-1}_{\alpha\beta}(z_1, z_2) \times C_{\alpha\gamma\delta}(z_1 - z_3)\, u^\beta(z_2)\, u^\gamma(z_3)\, u^\delta(z_3) \tag{37}$$

$$(Q(u), A^{-1}Q(u)) \equiv \int \cdots \int dz_1 \cdots dz_4\, A^{-1}_{\alpha\beta}(z_1, z_2) \times C_{\alpha\gamma\delta}(z_1 - z_3)\, C_{\beta\mu\nu}(z_2 - z_4) \times u^\gamma(z_3)\, u^\delta(z_3)\, u^\mu(z_4)\, u^\nu(z_4) \tag{38}$$

Consequently, from (35)

$$Z(\eta) = \int e^{i(\eta, u) + V_\lambda(u)}\, d\mu_G(u) \tag{39}$$

where $V_\lambda(u)$ is the exponent in expression (35). The relations (35) and (39) are exact but they involve $A^{-1}_{\alpha\beta}(z_1, z_2)$ and (39) is a nontrivial integration over the Gaussian measure μ_G. Formally, one may also write (39) as

$$Z(\eta) = e^{V_\lambda(\delta/i\delta\eta)} e^{-(\eta, A\eta)/2} \tag{40}$$

or

$$Z(\eta) \propto \left\{ \exp\left[-\frac{1}{2}\left(\frac{\delta}{i\delta\eta}, A^{-1}\frac{\delta}{i\delta\eta} \right) \right] \right\} W_\lambda(\eta) \tag{41}$$

where $W_\lambda(\eta)$ is the (inverse) Fourier transform of $\exp\{V_\lambda(\eta)\}$, which is not explicitly known. Expressions (40) and (41) correspond to "weak" and "strong" coupling expansions of $Z(\eta)$, respectively.[10] Note that the relation (40), or its general version

$$Z(\eta) = \left(\frac{d\mu}{d\mu_0}\left(\frac{\delta}{i\delta\eta} \right) Z_0 \right)(\eta) \tag{42}$$

mathematically is very different from (23); the operator in (42) depends on the prescription of $Z_0(\eta)$, as opposed to the generic nature of $\hat{S}$ in (23), which is valid for all admissible $Z_0(\eta)$'s. Here we will use the expression (23) for $Z_0(\eta) \equiv Z_G(\eta)$ as

$$Z(\eta) = e_L^{i\lambda(Q(\delta/i\delta\eta), \eta)} e^{-(\eta, A\eta)/2}, \tag{43}$$

a quantity that does not involve $A^{-1}_{\alpha\beta}(z_1, z_2)$ or a functional integration, and express the first-order terms in a corresponding λ expansion of the n-point space–time probability density function P_n of $u(z)$. The P_n is simply the (ordinary) Fourier transform of a particular restriction ϕ_n of $Z(\eta)$, namely

$$\begin{aligned} P_n(u_1,..., u_n; z_1, \alpha_1,..., z_n, \alpha_n) \\ = (2\pi)^{-n} \int \cdots \int dy_1 \cdots dy_n \exp\left(-i \sum_{k=1}^{n} y_k u_k\right) \\ \times \phi_n(y_1,..., y_n; z_1, \alpha_1,..., z_n, \alpha_n) \end{aligned} \tag{44}$$

with

$$\phi_n(y_1,..., y_n; z_1, \alpha_1,..., z_n, \alpha_n) = Z(\eta)\,|_{\eta^\alpha = \sum_k y_k \delta_{\alpha\alpha_k} \delta(z - z_k)} \tag{45}$$

in which $z_k \equiv (x_k, t_k) \in R^3 \times [t_0, \infty)$, u_k and $y_k \in R$, and $\alpha_k \in \{1, 2, 3\}$ for $1 \leqslant k \leqslant n$, $n = 1, 2,...$. Physically, P_n corresponds to the joint probability for a set of n statistical measurements of $u(z)$ in a turbulent flow field such that the kth measurement consists of the measurement of the α_k component of $u(z)$ at time t_k and position x_k, that is,

$$\begin{aligned} P_n(u_1,..., u_n; z_1, \alpha_1,..., z_n, \alpha_n)\, du_1 \cdots du_n \\ = \text{Prob} \left\{ \bigcap_{k=1}^{n} u_k < u^{\alpha_k}(z_k) < u_k + du_k \right\} \end{aligned} \tag{46}$$

Up to first order in λ, from (43) [or, equivalently, from the simpler relations (31) and (32)], we obtain

$$\begin{aligned} Z(\eta) = e^{-(\eta, A\eta)/2} \Big\{ 1 + i\lambda \iint dz_1\, dz_2\, C_{\alpha\beta\gamma}(z_1 - z_2)\, A_{\beta\gamma}(z_2, z_2)\, \eta^\alpha(z_1) \\ - i\lambda \int \cdots \int dz_1 \cdots dz_4\, C_{\alpha\beta\gamma}(z_1 - z_2)\, A_{\beta\delta}(z_2, z_3) \\ \times A_{\gamma\varepsilon}(z_2, z_4)\, \eta^\alpha(z_1)\, \eta^\delta(z_3)\, \eta^\varepsilon(z_4) \\ + O(\lambda^2) \Big\} \end{aligned} \tag{47}$$

The substitution of this last relation into (45) and the relation (44) directly give

$$P_n(u_1, ..., u_n; z_1, \alpha_1, ..., z_n, \alpha_n)$$
$$= P_G - \lambda \left\{ \sum_k B_{\alpha_k}(z_k) \frac{\partial P_G}{\partial u_k} + \sum_{k,l,m} B_{\alpha_k \alpha_l \alpha_m}(z_k, z_l, z_m) \frac{\partial^3 P_G}{\partial u_k \partial u_l \partial u_m} \right\} + O(\lambda^2) \tag{48}$$

where

$$B_\alpha(z_1) = \int dz\, C_{\alpha\beta\gamma}(z_1 - z)\, A_{\beta\gamma}(z, z) \tag{49}$$

$$B_{\alpha\beta\gamma}(z_1, z_2, z_3) = \int dz\, C_{\alpha\delta\varepsilon}(z_1 - z)\, A_{\beta\delta}(z_2, z)\, A_{\gamma\varepsilon}(z_3, z) \tag{50}$$

and P_G is the multivariate Gaussian probability density function

$$P_G = (2\pi)^{-n/2} (\det A)^{-1/2} \exp\left(-\frac{1}{2} \sum_{k,l} A^{-1}_{\alpha_k \alpha_l} u_k u_l \right) \tag{51}$$

which corresponds to the n-point characteristic function

$$\phi_G = \exp\left\{ -\frac{1}{2} \sum_{k,l} A_{\alpha_k \alpha_l}(z_k, z_l)\, y_k\, y_l \right\} \tag{52}$$

Similarly, the coefficient of λ for the moments (correlation functions) of $u(z)$ follow from (47) by using

$$\langle u^{\alpha_1}(z_1) \cdots u^{\alpha_n}(z_n) \rangle = (-i)^n \left. \frac{\delta^n Z(\eta)}{\delta\eta^{\alpha_1}(z_1) \cdots \delta\eta^{\alpha_n}(z_n)} \right|_{\eta=0} \tag{53}$$

This leads to a result which is identical to the one generated by the usual iterative approximation of Eq. (11).

Finally, for the space characteristic functional $\Phi(\theta, \tau) = Z(\theta\delta(t - \tau))$ $(t_0 < \tau < \infty)$ of $u(z)$, which carries information about the simultaneous statistics only, one can write

$$\Phi(\theta, \tau) = e^{-[\theta, A\theta]/2} \left\{ 1 + i\lambda \iint dx_1\, dz_2\, C_{\alpha\beta\gamma}(x_1 - x_2, \tau - t_2)\, A_{\beta\gamma}(z_2, z_2)\, \theta^\alpha(x_1) \right.$$
$$- i\lambda \int \cdots \int dx_1\, dx_3\, dx_4\, dz_2$$
$$\times C_{\alpha\beta\gamma}(x_1 - x_2, \tau - t_2)\, A_{\beta\delta}(z_2, x_3, \tau)\, A_{\gamma\varepsilon}(z_2, x_4, \tau)$$
$$\left. \times \theta^\alpha(x_1)\, \theta^\delta(x_3)\, \theta^\varepsilon(x_4) + O(\lambda^2) \right\} \tag{54}$$

where

$$[\theta, A\theta] \equiv \iint dx_1\, dx_2\, A_{\alpha\beta}(x_1, \tau, x_2, \tau)\, \theta^{\alpha}(x_1)\, \theta^{\beta}(x_2)$$

4. CONCLUDING REMARKS

Significant advances have been made in the last 15 years in the rigorous functional approach to the statistical theory of fully developed fluid turbulence[11–18] as well as in more formal exact treatments of Hopf's equation.[19] Almost all of these works involve a "space" framework in which the Navier–Stokes equation and its statistical version, the Hopf equation, are regarded as evolution equations in corresponding abstract spaces; the space–time approach is rarely employed.[20] Clearly, a complete statistical description of fluid turbulence requires a space–time point of view, since the actual fluid flows demonstrate both temporal and spatial randomness, and the functional Liouville equation associated with the Navier–Stokes dynamics cannot directly account for nonsimultaneous statistics. The formal unitary operator technique presented here is possible only within the space–time framework because it allows one to utilize $\mathrm{Tr}(\delta Q/\partial u)^n = 0$ for $n \geqslant 1$, which leads to $\det(T) = 1$.

Within this space-time operator approach the incorporation of random external forces is straightforward without restrictions on their statistics. In the evolutionary functional differential equation approach only the Gaussian external forces with delta correlations in time give a closed single equation,[21] while Gaussian forces with arbitrary correlations or arbitrary (non-Gaussian) forces lead to difficulties. The series representation (24) is also more transparent than the corresponding functional integral representation in assessing the effects of nonlinear interaction term $Q(u; z)$ on the prescribed statistics $Z_0(\eta)$, and it yields the explicit rules for a graphical analysis of $Z(\eta)$.

REFERENCES

1. E. Hopf, *J. Rat. Mech. Anal.* **1**:87 (1952).
2. G. Rosen, *Phys. Fluids* **3**:519 (1960).
3. G. Rosen, *J. Math. Phys.* **12**:81 (1971).
4. E. Hopf and E. W. Titt, *J. Rat. Mech. Anal.* **2**:587 (1953).
5. R. M. Lewis and R. H. Kraichnan, *Commun. Pure Appl. Math.* **15**:397 (1962).
6. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. 2 (MIT Press, Cambridge, Massachusetts, 1975).
7. G. K. Batchelor, *Theory of Homogeneous Turbulence* (Cambridge University Press, London, 1953).
8. G. Rosen, *Phys. Fluids* **13**:2891 (1970).

9. R. H. Kraichnan, in *Dynamics of Fluids and Plasmas*, S. I. Pai, ed. (Academic Press, New York, 1966), pp. 239–255.
10. C. M. Bender, F. Cooper, G. S. Guralnik, and D. H. Sharp, *Phys. Rev. D* **19**:1865 (1979).
11. C. Foias, *Rend. Sem. Mat. Univ. Padova* **48**:219 (1972); **49**:9 (1973).
12. A. Bensoussan and R. Temam, *J. Func. Anal.* **13**:195 (1973).
13. C. Foias, *Russ. Math. Surv.* **29**:293 (1975).
14. M. I. Vishik, *Sov. Math. Dokl.* **16**:1126 (1975).
15. A. M. Versik and O. A. Ladyzenskaya, *Sov. Math. Dokl.* **17**:18 (1976).
16. M. I. Vishik and A. V. Fursikov, *Ann. Sc. Norm. Super. Pisa* **IV**:531 (1977); *J. Math. Pure Appl.* **56**:85 (1977); *Sib. Math. J.* **19**:710 (1978).
17. C. Foias and R. Temam, *Ind. Univ. Math. J.* **29**:913 (1980).
18. C. Foias, O. P. Manley, and R. Temam, *Phys. Fluids* **30**:2007 (1987).
19. G. Rosen, *J. Math. Phys.* **23**:2582 (1983); **24**:193 (1983); *Phys. Rev. A* **32**:2549 (1985).
20. A. A. Arseniev, *Sov. Math. Dokl.* **16**:1422 (1975).
21. E. A. Novikov, *Sov. Phys. JETP* **20**:1290 (1965).